

EXTENSION THEOREM OF WHITNEY TYPE FOR $\mathcal{S}(\mathbb{R}_+^d)$ BY THE USE OF THE KERNEL THEOREM

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ABSTRACT. We study the expansions of the elements in $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$ with respect to the Laguerre orthonormal basis, extending the result of M. Guillemont-Teissier [4] in the case $d = 1$. As a consequence, we obtain the Schwartz kernel theorem for $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$ and the extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}_+^d)$.

1. INTRODUCTION

We denote by \mathbb{R}_+^d the set $(0, \infty)^d$ and by $\overline{\mathbb{R}_+^d}$ its closure, i.e. $[0, \infty)^d$. We will consider the space $\mathcal{S}(\mathbb{R}_+^d)$ which consists of all $f \in \mathcal{C}^\infty(\mathbb{R}_+^d)$ such that all derivatives $D^p f$, $p \in \mathbb{N}_0^d$, extend to continuous functions on $\overline{\mathbb{R}_+^d}$ and

$$\sup_{x \in \mathbb{R}_+^d} x^k |D^p f(x)| < \infty, \forall k, p \in \mathbb{N}_0^d.$$

With this system of seminorms $\mathcal{S}(\mathbb{R}_+^d)$ becomes an (F) -space.

The results concerning the extension of a smooth function or a function of class \mathcal{C}^k out of some region and various reformulation of such problems are called extension theorems of Whitney type. One can see Whitney [11], Seeley [8] and Hörmander [3, Theorem 2.3.6, p. 48]. Here we deal with a problem of extension of a function from $\mathcal{S}(\mathbb{R}_+^d)$ onto $\mathcal{S}(\mathbb{R}^d)$. Theorem 4.3 is the main result of the paper. For the purpose of this theorem we prove the Schwartz kernel theorem for $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$, Theorem 4.2.

Recall, for $n = 0, 1, 2, \dots$ the functions

$$L_n(x) = \frac{e^x}{n!} \left(\frac{d}{dx} \right)^n (e^{-x} x^n), \quad x > 0$$

are the Laguerre polynomials and $\mathcal{L}_n(x) = L_n(x) e^{-\frac{x}{2}}$ are the Laguerre functions; $\{\mathcal{L}_n(x), n = 0, 1, \dots\}$ is an orthonormal basis for $L^2(0, \infty)$ ([10] p.108).

The problem of expanding the elements of $\mathcal{S}'(\mathbb{R}_+)$ with respect to the Laguerre orthonormal basis has been treated by Guillemont-Teissier in [4] and Duran in [1]:

If $T \in \mathcal{S}'(\mathbb{R}_+)$ and $a_n = \langle T, \mathcal{L}_n(x) \rangle$ then $T = \sum_{n=0}^{\infty} a_n \mathcal{L}_n(x)$ and $\{a_n\}_{n=0}^{\infty}$ decreases slowly. Conversely, if $\{a_n\}_{n=0}^{\infty}$ decreases slowly, then there exists $T \in \mathcal{S}'(\mathbb{R}_+)$ such that $T = \sum_{n=0}^{\infty} a_n \mathcal{L}_n(x)$.

The works [7], [12] and [13] contain expansions of the same kind as in [4] and [1].

The novelty of this paper is the extension of the results of [4] for the d dimensional case. This leads to the Schwartz kernel theorem (Theorem 4.2) which states that there is one-to-one correspondence between elements from $\mathcal{S}'(\mathbb{R}_+^{m+n})$ in two sets of variables x and y and the continuous linear mappings of $(\mathcal{S}(\mathbb{R}_+^m))_y$ into $(\mathcal{S}'(\mathbb{R}_+^n))_x$. As a consequence of Theorem 4.3 we explain the convolution in $\mathcal{S}'(\mathbb{R}_+^d)$ in the last remark.

The plan of the paper is as follows. We recall in section 3 some properties of the Laguerre series and we prove the convergence of the Laguerre series in $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$. In section 4 we state the Schwartz's kernel theorem for $\mathcal{S}(\mathbb{R}_+^d)$ and we prove the extension theorem of Whitney type for $\mathcal{S}(\mathbb{R}_+^d)$.

2. NOTATION

We use the standard multi-index notation. Given $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\alpha| = \sum_{i=1}^d \alpha_i$, $x^\alpha = (x_1, \dots, x_d)^{(\alpha_1, \dots, \alpha_d)} = \prod_{i=1}^d x_i^{\alpha_i}$, $D^\alpha = \prod_{i=1}^d \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$ for the partial derivative and $X^\alpha f(x) = x^\alpha f(x)$ for the multiplication operator. For $x \in \mathbb{R}^d$, $|x|$ stands for the standard Euclidean norm in \mathbb{R}^d .

Let s be the space of rapidly decreasing sequences

$$\{a_n\}_{n \in \mathbb{N}_0^d} \in s \Leftrightarrow \sum_{n \in \mathbb{N}_0^d} |a_n|^2 n^{2k} < \infty, \quad \forall k \in \mathbb{N}.$$

Then s' stands for the strong dual of s , the space of slowly increasing sequences

$$\{a_n\}_{n \in \mathbb{N}_0^d} \in s' \Leftrightarrow \sum_{n \in \mathbb{N}_0^d} |a_n|^2 n^{-2k} < \infty, \quad \exists k \in \mathbb{N}.$$

3. LAGUERRE SERIES

The d -dimensional Laguerre functions

$$\mathcal{L}_n(x) = \mathcal{L}_{n_1}(x_1) \dots \mathcal{L}_{n_d}(x_d) = \prod_{i=1}^d \mathcal{L}_{n_i}(x_i)$$

form an orthonormal basis for $L^2(\mathbb{R}_+^d)$ and are the eigenfunctions of the Laguerre operator $E = (D_1(x_1 D_1) - \frac{x_1}{4}) \dots (D_d(x_d D_d) - \frac{x_d}{4})$, $E : \mathcal{S}(\mathbb{R}_+^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$

$$\mathcal{L}_n(x) \rightarrow E(\mathcal{L}_n(x)) = \prod_{i=1}^d -(n_i + \frac{1}{2}) \mathcal{L}_n(x).$$

Notice that E is a self-adjoint operator, i.e.

$$\langle Ef, g \rangle = \langle f, Eg \rangle, \quad f, g \in \text{dom}(E) = \{f \in L^2(\mathbb{R}_+^d); Ef \in L^2(\mathbb{R}_+^d)\}.$$

For $f \in \mathcal{S}(\mathbb{R}_+^d)$ we define the n -th Laguerre coefficient by $a_n = \int_{\mathbb{R}_+^d} f(x) \mathcal{L}_n(x) dx$. The Laguerre series of the function $f \in \mathcal{S}(\mathbb{R}_+^d)$ is $\sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n(x)$.

In [4], p. 547 the following bound on the one-dimensional Laguerre functions is obtained:

$$\left| x^k \left(\frac{d}{dx} \right)^p \mathcal{L}_n(x) \right| \leq C_{p,k} (n+1)^{p+k}, \quad x \geq 0, \quad n, p, k \geq 0.$$

Finding the bound on the d -dimensional Laguerre functions involves not complicated calculation. Hence:

$$|x^k D^p \mathcal{L}_n(x)| \leq C_{p,k} \prod_{i=1}^d (n_i + 1)^{p_i + k_i}, \quad x \in \mathbb{R}_+^d, \quad n, p, k \in \mathbb{N}_0^d. \quad (1)$$

3.1. Convergence of the Laguerre series in $\mathcal{S}(\mathbb{R}_+^d)$.

Theorem 3.1. *For $f \in \mathcal{S}(\mathbb{R}_+^d)$ let $a_n(f) = \int_{\mathbb{R}_+^d} f(x) \mathcal{L}_n(x) dx$. Then $f = \sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{L}_n$ and the series converges absolutely in $\mathcal{S}(\mathbb{R}_+^d)$. Moreover the mapping $\iota : \mathcal{S}(\mathbb{R}_+^d) \rightarrow s$, $\iota(f) = \{a_n(f)\}_{n \in \mathbb{N}_0^d}$ is a topological isomorphism.*

Proof. For $f \in \mathcal{S}(\mathbb{R}_+^d)$

$$a_n(Ef) = \langle Ef, \mathcal{L}_n \rangle = \langle f, E(\mathcal{L}_n) \rangle = a_n(f) (-1)^d \prod_{i=1}^d \left(n_i + \frac{1}{2} \right).$$

Moreover,

$$a_n(E^p f) = a_n(f) \prod_{i=1}^d (-1)^{p_i} \left(n_i + \frac{1}{2} \right)^{p_i}$$

for any $p \in \mathbb{N}^d$. As $E^p f \in \mathcal{S}(\mathbb{R}_+^d) \subset L^2(\mathbb{R}_+^d)$, we have

$$\sum_{n \in \mathbb{N}_0^d} |a_n(f)|^2 \prod_{i=1}^d \left(n_i + \frac{1}{2} \right)^{2p_i} < \infty, \quad \text{for every } p \in \mathbb{N}_0^d,$$

i.e. $\{a_n(f)\}_{n \in \mathbb{N}_0^d} \in s$. Clearly $f = \sum_{n \in \mathbb{N}_0^d} a_n(f) \mathcal{L}_n$ as elements of $L^2(\mathbb{R}_+^d)$. By (1), we obtain

$$\sum_{n \in \mathbb{N}_0^d} |x^k D^p (a_n(f) \mathcal{L}_n(x))| \leq C_{p,k} \sum_{n \in \mathbb{N}_0^d} |a_n(f)| \prod_{i=1}^d (n_i + 1)^{p_i + k_i} < \infty \quad (2)$$

which yields the absolute convergence of the series in $\mathcal{S}(\mathbb{R}_+^d)$.

To prove that ι is topological isomorphism, first observe that by the above consideration it is well defined and it is clearly an injection. Let $\{a_n\}_{n \in \mathbb{N}_0^d} \in s$. Define $f = \sum_{n \in \mathbb{N}_0^d} a_n \mathcal{L}_n \in L^2(\mathbb{R}_+^d)$. Now (2) proves that this series converges in $\mathcal{S}(\mathbb{R}_+^d)$, hence $f \in \mathcal{S}(\mathbb{R}_+^d)$. Thus ι is bijective. Observe that, (2) proves that ι^{-1} is continuous. Since $\mathcal{S}(\mathbb{R}_+^d)$ and s are (F) -spaces, the open mapping theorem proves that ι is topological isomorphism. \square

3.2. Convergence of the Laguerre series in $\mathcal{S}'(\mathbb{R}_+^d)$.

Theorem 3.2. *For $T \in \mathcal{S}'(\mathbb{R}_+^d)$, let $b_n(T) = \langle T, \mathcal{L}_n \rangle$. Then $T = \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n$ and $\{b_n(T)\}_{n \in \mathbb{N}_0^d} \in s'$. The series converges absolutely in $\mathcal{S}'(\mathbb{R}_+^d)$. Conversely, if $\{b_n\}_{n \in \mathbb{N}_0^d} \in s'$, then there exists $T \in \mathcal{S}'(\mathbb{R}_+^d)$ such that $T = \sum_{n \in \mathbb{N}_0^d} b_n \mathcal{L}_n$. As a consequence, $\mathcal{S}'(\mathbb{R}_+^d)$ is topologically isomorphic to s' .*

Proof. Let $\{b_n\}_{n \in \mathbb{N}_0^d} \in s'$. There exists $k \in \mathbb{N}$ such that $\sum_{n \in \mathbb{N}_0^d} |b_n|^2 (|n| + 1)^{-2k} < \infty$. For a bounded subset B of $\mathcal{S}(\mathbb{R}_+^d)$, Theorem 3.1 implies that there exists $C > 0$ such that

$$\sum_{n \in \mathbb{N}_0^d} |a_n(f)|^2 (|n| + 1)^{2k} \leq C, \quad \forall f \in B,$$

where we denote $\{a_n(f)\}_{n \in \mathbb{N}_0^d} = \iota(f)$. Observe that for arbitrary $q \in \mathbb{N}$ we have

$$\sum_{|n| \leq q} \sup_{f \in B} |\langle b_n \mathcal{L}_n, f \rangle| \leq \sup_{f \in B} \sum_{n \in \mathbb{N}_0^d} \sum_{m \in \mathbb{N}_0^d} |\langle b_n \mathcal{L}_n, a_m(f) \mathcal{L}_m \rangle| = \sup_{f \in B} \sum_{n \in \mathbb{N}_0^d} |b_n| |a_n(f)| \leq C',$$

i.e.

$$\sum_{n \in \mathbb{N}_0^d} \sup_{f \in B} |\langle b_n \mathcal{L}_n, f \rangle| < \infty,$$

hence $\sum_{n \in \mathbb{N}_0^d} b_n \mathcal{L}_n$ converges absolutely in $\mathcal{S}'(\mathbb{R}_+^d)$.

Let $T \in \mathcal{S}'(\mathbb{R}_+^d)$. Theorem 3.1 implies that ${}^t\iota : s' \rightarrow \mathcal{S}'(\mathbb{R}_+^d)$ is an isomorphism (${}^t\iota$ denotes the transpose of ι). Now, one easily verifies that $({}^t\iota)^{-1}T = \{b_n\}_{n \in \mathbb{N}_0^d}$, where $b_n(T) = \langle T, \mathcal{L}_n \rangle$. Observe that for $f \in \mathcal{S}(\mathbb{R}_+^d)$

$$\langle T, f \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) \langle T, \mathcal{L}_n \rangle = \sum_{n \in \mathbb{N}_0^d} a_n(f) b_n(T) = \left\langle \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n, f \right\rangle,$$

i.e. $T = \sum_{n \in \mathbb{N}_0^d} b_n(T) \mathcal{L}_n$. □

4. KERNEL THEOREM

The completions of the tensor product are denoted by $\hat{\otimes}_\epsilon$ and $\hat{\otimes}_\pi$ with respect to ϵ and π topologies. If they are equal we drop the subindex.

Proposition 4.1. *The spaces $\mathcal{S}(\mathbb{R}_+^d)$ and $\mathcal{S}'(\mathbb{R}_+^d)$ are nuclear.*

Proof. Since s is nuclear Theorem 3.1 implies that $\mathcal{S}(\mathbb{R}_+^d)$ is also nuclear. Now $\mathcal{S}'(\mathbb{R}_+^d)$ is nuclear as the strong dual of a nuclear (F) -space. □

Theorem 4.2. *The following canonical isomorphisms hold:*

$$\mathcal{S}(\mathbb{R}_+^m) \hat{\otimes} \mathcal{S}(\mathbb{R}_+^n) \cong \mathcal{S}(\mathbb{R}_+^{m+n}), \quad \mathcal{S}'(\mathbb{R}_+^m) \hat{\otimes} \mathcal{S}'(\mathbb{R}_+^n) \cong \mathcal{S}'(\mathbb{R}_+^{m+n}).$$

Proof. The second isomorphism follows from the first since $\mathcal{S}(\mathbb{R}_+^d)$ is a nuclear (F) -space. Thus it is enough to prove the first isomorphism.

Step 1: From Theorem 3.1 follows that $\mathcal{S}(\mathbb{R}_+^m) \otimes \mathcal{S}(\mathbb{R}_+^n)$ is dense in $\mathcal{S}(\mathbb{R}_+^{m+n})$. It suffices to show that the latter induces on the former the topology $\pi = \epsilon$ (the π and the ϵ topologies are the same because $\mathcal{S}(\mathbb{R}_+^d)$ is nuclear). Since the bilinear mapping $(f, g) \mapsto f \otimes g$ of $\mathcal{S}(\mathbb{R}_+^m) \times \mathcal{S}(\mathbb{R}_+^n)$ into $\mathcal{S}(\mathbb{R}_+^{m+n})$ is separately continuous it follows that it is continuous ($\mathcal{S}(\mathbb{R}_+^m)$ and $\mathcal{S}(\mathbb{R}_+^n)$ are (F) -spaces). The continuity of this bilinear mapping proves that the inclusion $\mathcal{S}(\mathbb{R}_+^m) \otimes_\pi \mathcal{S}(\mathbb{R}_+^n) \rightarrow \mathcal{S}(\mathbb{R}_+^{m+n})$ is continuous, hence the topology π is stronger than the induced one from $\mathcal{S}(\mathbb{R}_+^{m+n})$ onto $\mathcal{S}(\mathbb{R}_+^m) \otimes \mathcal{S}(\mathbb{R}_+^n)$.

Step 2: Let A' and B' be equicontinuous subsets of $\mathcal{S}'(\mathbb{R}_+^m)$ and $\mathcal{S}'(\mathbb{R}_+^n)$, respectively. There exist $C > 0$ and $j, l \in \mathbb{N}$ such that

$$\sup_{T \in A'} |\langle T, \varphi \rangle| \leq C \|\varphi\|_{j,l} \quad \text{and} \quad \sup_{F \in B'} |\langle F, \psi \rangle| \leq C \|\psi\|_{j,l},$$

where

$$\|f\|_{j,l} = \sup_{\substack{|k| \leq j \\ |p| \leq l}} \sup_{x \in \mathbb{R}_+^d} |x^k D^p f(x)| < \infty. \quad (3)$$

For all $T \in A'$ and $F \in B'$ we have

$$\begin{aligned} |\langle T_x \otimes F_y, \chi(x, y) \rangle| &= |\langle F_y, \langle T_x, \chi(x, y) \rangle \rangle| \leq C \sup_{\substack{|k| \leq j \\ |p| \leq l}} \sup_{y \in \mathbb{R}_+^n} |y^k \langle T_x, D_y^p \chi(x, y) \rangle| \\ &\leq C^2 \sup_{\substack{|k| \leq j \\ |p| \leq l}} \sup_{\substack{|k'| \leq j \\ |p'| \leq l}} \sup_{\substack{x \in \mathbb{R}_+^m \\ y \in \mathbb{R}_+^n}} |x^{k'} y^k D_x^{p'} D_y^p \chi(x, y)| \\ &\leq C^2 \|\chi(x, y)\|_{(k', k), (p', p)}, \quad \forall \chi \in \mathcal{S}(\mathbb{R}_+^m) \otimes \mathcal{S}(\mathbb{R}_+^n). \end{aligned}$$

It follows that the ϵ topology on $\mathcal{S}(\mathbb{R}_+^m) \otimes \mathcal{S}(\mathbb{R}_+^n)$ is weaker than the induced one from $\mathcal{S}(\mathbb{R}_+^{m+n})$. \square

As a consequence of this theorem we have the following important

Theorem 4.3. *The restriction mapping $f \mapsto f|_{\mathbb{R}_+^d}$, $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$ is a topological homomorphism onto.*

The space $\mathcal{S}(\mathbb{R}_+^d)$ is topologically isomorphic to the quotient space $\mathcal{S}(\mathbb{R}^d)/N$, where $N = \{f \in \mathcal{S}(\mathbb{R}^d) \mid \text{supp } f \subseteq \mathbb{R}^d \setminus \mathbb{R}_+^d\}$. Consequently, $\mathcal{S}'(\mathbb{R}_+^d)$ can be identified with the closed subspace of $\mathcal{S}'(\mathbb{R}^d)$ which consists of all tempered distributions with support in $\overline{\mathbb{R}_+^d}$.

Proof. Obviously, the restriction mapping $f \mapsto f|_{\mathbb{R}_+^d}$, $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}(\mathbb{R}_+^d)$ is continuous. We prove its surjectivity by induction on d . For clarity, denote the d -dimensional restriction by R_d . For $d = 1$, the surjectivity of R_1 is proved in [1, p. 168]. Assume that R_d is surjective. By the open mapping theorem, R_d and R_1 are topological homomorphisms onto since all the underlying spaces are (F) -spaces. By the above theorem $R_d \hat{\otimes}_\pi R_1$ is continuous mapping from $\mathcal{S}(\mathbb{R}^{d+1})$ to $\mathcal{S}(\mathbb{R}_+^{d+1})$ ($\mathcal{S}(\mathbb{R}^d) \hat{\otimes} \mathcal{S}(\mathbb{R}) \cong \mathcal{S}(\mathbb{R}^{d+1})$ by the Schwartz kernel theorem). Clearly $R_d \hat{\otimes}_\pi R_1 = R_{d+1}$. As $\mathcal{S}(\mathbb{R}^{d+1})$ and $\mathcal{S}(\mathbb{R}_+^{d+1})$ are (F) -spaces [6, Theorem 7, p. 189] implies that R_{d+1} is also surjective.

The surjectivity of the restriction mapping together with the open mapping theorem implies that it is homomorphism. Clearly N is closed subspace of $\mathcal{S}(\mathbb{R}^d)$ and $\ker R_d = N$. Thus R_d induces natural topological isomorphism between $\mathcal{S}(\mathbb{R}^d)/N$ and $\mathcal{S}(\mathbb{R}_+^d)$. Hence $(\mathcal{S}(\mathbb{R}^d)/N)'_b$ is topologically isomorphic to $\mathcal{S}'(\mathbb{R}_+^d)$ (the index b stands for the strong dual topology). Since $\mathcal{S}(\mathbb{R}^d)$ is an (FS) -space, [5, Theorem A.6.5, p. 255] implies that $(\mathcal{S}(\mathbb{R}^d)/N)'_b$ is topologically isomorphic to the closed subspace

$$N^\perp = \{T \in \mathcal{S}'(\mathbb{R}^d) \mid \langle T, f \rangle = 0, \forall f \in N\}$$

of $\mathcal{S}'(\mathbb{R}^d)$ which is exactly the subspace of all tempered distributions with support in $\overline{\mathbb{R}_+^d}$. \square

Given $f, g \in \mathcal{S}'(\mathbb{R}_+^d)$, Theorem 4.3 implies that we can consider them as elements of $\mathcal{S}'(\mathbb{R}^d)$ with support in $\overline{\mathbb{R}_+^d}$. Now, one easily verifies that for each $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(f(x) \otimes g(y))\varphi(x+y) \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d}),$$

hence the \mathcal{S}' -convolution of f and g exists (see [9, p. 26]). Also, if $\text{supp } \varphi \cap \overline{\mathbb{R}_+^d} = \emptyset$, then $(f(x) \otimes g(y))\varphi(x+y) = 0$, hence $\text{supp } f * g \subseteq \overline{\mathbb{R}_+^d}$, i.e. $f * g \in \mathcal{S}'(\mathbb{R}_+^d)$. Thus

$$\langle f * g, \varphi \rangle = \langle f(x) \otimes g(y), \varphi(x+y) \rangle, \varphi \in \mathcal{S}(\mathbb{R}_+^d)$$

(observe that the function $\varphi^\Delta(x, y) = \varphi(x+y)$ is an element of $\mathcal{S}(\mathbb{R}_+^{2d})$).

Remark 4.4. ([1], Remark 3.7 for $d=1$) Let us show that $\mathcal{S}'(\mathbb{R}_+^d)$ is a convolution algebra. Given $f, g \in \mathcal{S}'(\mathbb{R}_+^d)$, we compute the n -th Laguerre coefficient of $f * g$ if $a_n = \langle f, \mathcal{L}_n \rangle$ and $b_n = \langle g, \mathcal{L}_n \rangle$ then

$$\langle f * g, \mathcal{L}_n(t) \rangle = \langle f(x) \otimes g(y), \mathcal{L}_n(x+y) \rangle.$$

Now, $L_n^1(x+y) = \sum_{k=0}^n L_{n-k}(x)L_k(y)$ and $L_n(t) = L_n^1(t) - L_{n-1}^1(t)$ (see [2], p. 192), where $L_n^1(x) = \sum_{k=0}^n \binom{n+1}{n-k} ((-x)^k/k!)$. In order to simplify the proof, we consider the case $d = 2$. Then

$$\begin{aligned} \langle f * g, \mathcal{L}_n(t) \rangle &= \langle f(x) \otimes g(y), \prod_{i=1}^2 (\mathcal{L}_{n_i}^1(x_i + y_i) - \mathcal{L}_{n_i-1}^1(x_i + y_i)) \rangle \\ &= \langle f(x) \otimes g(y), \prod_{i=1}^2 \left(\sum_{k_i=0}^{n_i} \mathcal{L}_{n_i-k_i}(x_i) \mathcal{L}_{k_i}(y_i) - \sum_{k_i=0}^{n_i-1} \mathcal{L}_{n_i-k_i-1}(x_i) \mathcal{L}_{k_i}(y_i) \right) \rangle \\ &= \langle f(x)g(y), \sum_{k \leq (n_1, n_2)} \mathcal{L}_{(n_1, n_2)-k}(x) \mathcal{L}_k(y) - \sum_{k \leq (n_1-1, n_2)} \mathcal{L}_{(n_1-1, n_2)-k}(x) \mathcal{L}_k(y) \\ &\quad - \sum_{k \leq (n_1, n_2-1)} \mathcal{L}_{(n_1, n_2-1)-k}(x) \mathcal{L}_k(y) + \sum_{k \leq (n_1-1, n_2-1)} \mathcal{L}_{(n_1-1, n_2-1)-k}(x) \mathcal{L}_k(y) \rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \leq (n_1, n_2)} a_{(n_1, n_2) - k} b_k - \sum_{k \leq (n_1 - 1, n_2)} a_{(n_1 - 1, n_2) - k} b_k \\
&\quad - \sum_{k \leq (n_1, n_2 - 1)} a_{(n_1, n_2 - 1) - k} b_k + \sum_{k \leq (n_1 - 1, n_2 - 1)} a_{(n_1 - 1, n_2 - 1) - k} b_k,
\end{aligned}$$

where a_n or b_n equals zero if some component of the subindex n is less than zero. It is easy to verify that if $(a_n)_{n \in \mathbb{N}^2} \in s'$ and $(b_n)_{n \in \mathbb{N}^2} \in s'$ then $\langle f * g, \mathcal{L}_n(t) \rangle \in s'$.

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